Rational symmetric functions from the Izergin–Korepin 19-vertex model

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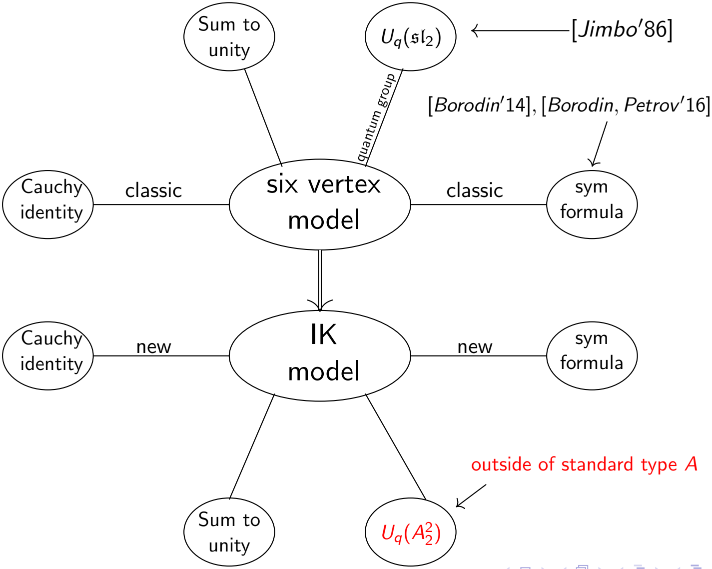
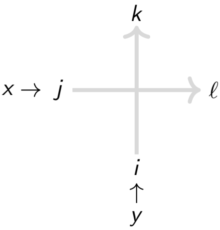
Motivations



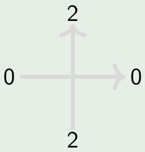
Izergin–Korepin model

Izergin–Korepin model [Izergin,Korepin'81] has weights

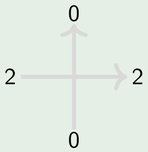
$W_{y/x}(i,j;k,\ell) :=$ 

where $i,j,k,\ell \in \{0,1,2\}$.

Example



$= \frac{q^4(x-y)(x-qy)}{(x-q^2y)(x-q^3y)}$



$= \frac{(x-y)(x-qy)}{(x-q^2y)(x-q^3y)}$

Izergin–Korepin model

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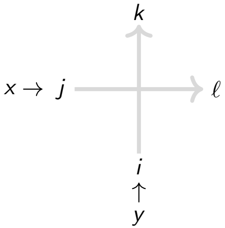
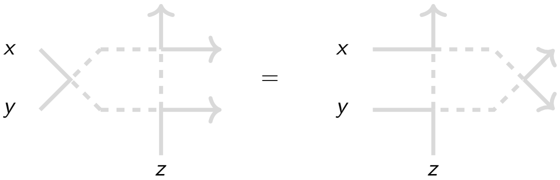


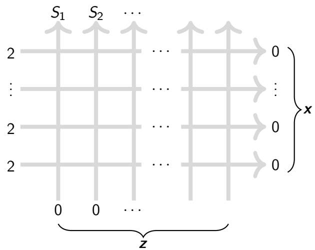
where $i,j,k,\ell \in \{0,1,2\}$.

The Yang–Baxter equation is



Rational function (Izergin–Korepin model)

The rational functions $F_S(\boldsymbol{x};\boldsymbol{z})$(left panel) and $G_S(\boldsymbol{y};\boldsymbol{z})$(right panel) are



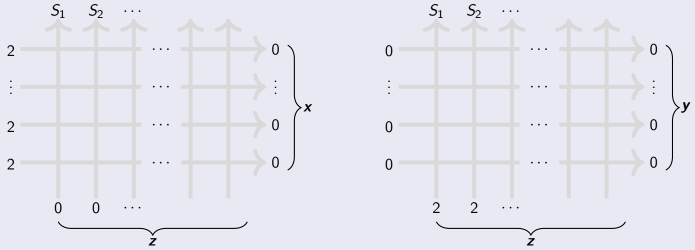
where $S = (S_1, S_2, \dots)$ and $S_i \in \{0, 1, 2\}$.

Example:

Rational function (Izergin–Korepin model)

Definition

The rational functions $F_S(\boldsymbol{x};\boldsymbol{z})$(left panel) and $G_S(\boldsymbol{y};\boldsymbol{z})$(right panel) are



where $S = (S_1, S_2, \dots)$ and $S_i \in \{0, 1, 2\}$.

Theorem

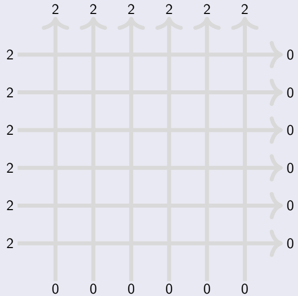
The partition functions $F_S(\boldsymbol{x};\boldsymbol{z})$ and $G_S(\boldsymbol{y};\boldsymbol{z})$ are symmetric in $\boldsymbol{x}$ and $\boldsymbol{y}$.

Domain–wall partition function (Izergin–Korepin model)

Definition

The domain–wall partition function [Garbali'15] is

$$F_{(2^N)}(\boldsymbol{x};\boldsymbol{z}) :=$$



Taking $\boldsymbol{z} \to \infty$, the partition function becomes

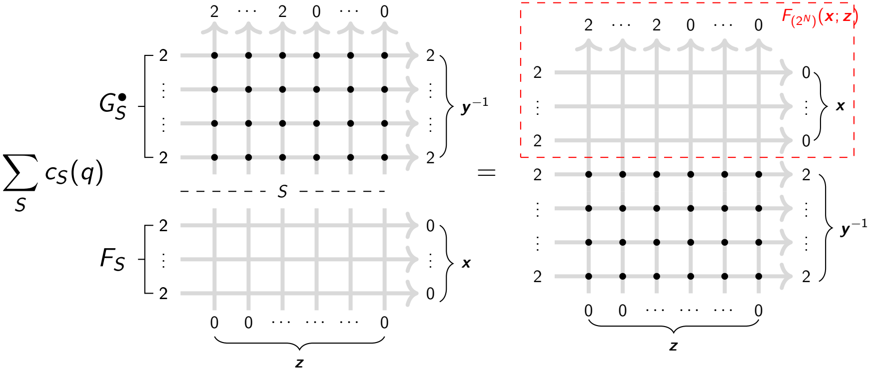
$$Z_N(q) := \lim_{\boldsymbol{z}\to\infty} F_{(2^N)}(\boldsymbol{x};\boldsymbol{z}) = \left(\prod_{i=1}^{N} 1 - q^{-2i}\right)\left(\prod_{j=1}^{N} 1 + q^{-2j+1}\right).$$

Cauchy identity (Izergin–Korepin model)

The Izergin–Korepin Cauchy identity is given by

$$\sum_S c_S(q) F_S(\boldsymbol{x};\boldsymbol{z}) G_S(\boldsymbol{y};(q^3\boldsymbol{z})^{-1}) = F_{(2^N)}(\boldsymbol{x};\boldsymbol{z}) \prod_{i,j} \frac{(1-q^2x_iy_j)(1-q^3x_iy_j)}{(1-x_iy_j)(1-qx_iy_j)}.$$

The pictorial presentation is



Stable rational function (Izergin–Korepin model)

Definition

The *stable rational functions* are defined by

$$H_S(\boldsymbol{x};\boldsymbol{z}) := \frac{1}{Z_{|\boldsymbol{u}|}(q)} \lim_{\boldsymbol{u}\to\infty} F_S(\boldsymbol{x};\boldsymbol{u}\cup\boldsymbol{z}).$$

Theorem

The rational function H_S satisfies the stability property

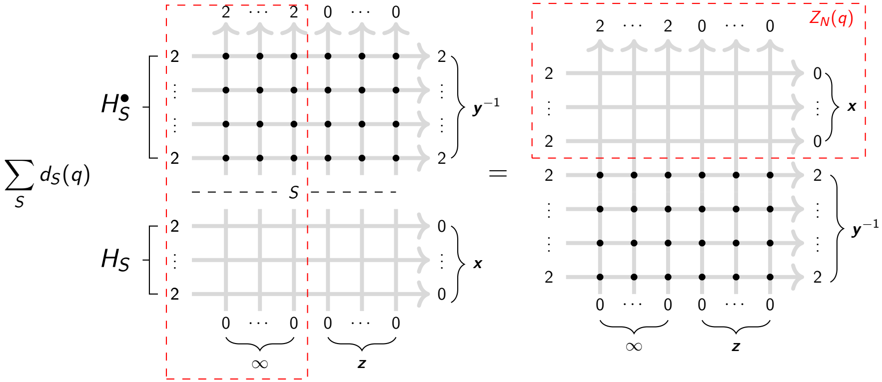
$$H_S(x_1, x_2, \ldots, x_N = \infty; \boldsymbol{z}) = \begin{cases} H_S(x_1, x_2, \ldots, x_{N-1}; \boldsymbol{z}) & |S| < 2N-2, \\ 0 & \textit{otherwise.} \end{cases}$$

Cauchy identity for stable functions

The Cauchy identity simplifies

$$\sum_S d_S(q) H_S(\boldsymbol{x};\boldsymbol{z}) H_S(\boldsymbol{y};(q^3\boldsymbol{z})^{-1}) = \prod_{i,j} \frac{(1-q^2x_iy_j)(1-q^3x_iy_j)}{(1-x_iy_j)(1-qx_iy_j)}.$$

The pictorial presentation is



Symmetrization formula (Izergin–Korepin model)

What is a symmetrization formula?

- An explicit form of $F_S(\boldsymbol{x};\boldsymbol{z})$
- Easy to compute and digest

For example,

Schur function

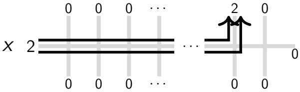
combinatorial formula

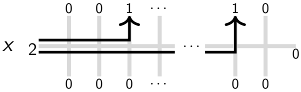
$$\sum_{T \in SSYT} wt_T$$

symmetrization formula

$$\sum_{\sigma \in \mathfrak{S}_n} \sigma \left\{ \prod_{i<j} \frac{x_i}{x_i - x_j} \prod_k x_k^{\lambda_k} \right\}$$

- One row partition functions





Symmetrization formula(Izergin–Korepin model)

$$F_S(\boldsymbol{x};\boldsymbol{z}) = \sum_{\sigma \in M_S} \Delta_\sigma(\boldsymbol{x};\boldsymbol{z}) \prod_i F_{\sigma(i)}(\boldsymbol{x};\boldsymbol{z})$$

2permutations

$S=(2^2)$, $\begin{pmatrix}2&0\\0&2\end{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}2&0\\0&2\end{pmatrix}$

$S=(2^3)$, $\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}\begin{pmatrix}2&0&0\\0&1&1\\0&1&1\end{pmatrix}\begin{pmatrix}2&0&0\\0&0&2\\0&2&0\end{pmatrix}$

$\begin{pmatrix}1&1&0\\1&1&0\\0&0&2\end{pmatrix}\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}$

$\begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix}\begin{pmatrix}1&1&0\\0&0&2\\1&1&0\end{pmatrix}\begin{pmatrix}0&2&0\\2&0&0\\0&0&2\end{pmatrix}$

$\begin{pmatrix}0&2&0\\1&0&1\\1&0&1\end{pmatrix}\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}\begin{pmatrix}0&2&0\\0&0&2\\2&0&0\end{pmatrix}$

$\vdots$

Symmetrization formula(Izergin–Korepin model)

$$F_S(\boldsymbol{x};\boldsymbol{z}) = \sum_{\sigma \in M_S} \Delta_\sigma(\boldsymbol{x};\boldsymbol{z}) \prod_i F_{\sigma(i)}(\boldsymbol{x};\boldsymbol{z})$$

2permutations | scattering factor | one row functions

$S=(2^2)$, $\begin{pmatrix}2&0\\0&2\end{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}2&0\\0&2\end{pmatrix}$

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$\begin{pmatrix}1&1&0\\1&1&0\\0&0&2\end{pmatrix}\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}$

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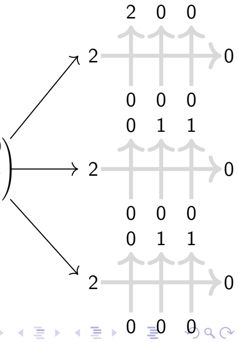
$\begin{pmatrix}0&2&0\\1&0&1\\1&0&1\end{pmatrix}\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix}\begin{pmatrix}0&2&0\\0&0&2\\2&0&0\end{pmatrix}$

$\vdots$

next page !

$\begin{pmatrix}2&0&0\\0&1&1\\0&1&1\end{pmatrix}$



Scattering factor

- $\Delta_\sigma(\boldsymbol{x};\boldsymbol{z})$ is a rational function
- $\Delta_\sigma(\boldsymbol{x};\boldsymbol{z})$ is a product of pairs of rows

List of all factors

$$\Delta_{V,U}(x_i,x_j;z)=\begin{cases}
\dfrac{(q^2x_i-x_j)(q^3x_i-x_j)}{(x_i-x_j)(qx_i-x_j)}, & \begin{array}{c||c|c|c} U & & 2 & \\ \hline V & 2 & & \end{array}, \begin{array}{c||c|c|c} U & & 1 & 1 \\ \hline V & 2 & & \end{array}, \begin{array}{c||c|c|c} U & & & 2 \\ \hline V & 1 & 1 & \end{array}, \begin{array}{c||c|c|c|c} U & & & 1 & 1 \\ \hline V & 1 & 1 & & \end{array}, \\
\dfrac{(x_i-q^2x_j)(x_i-q^3x_j)}{(x_i-x_j)(x_i-qx_j)}, & \begin{array}{c||c|c|c} U & 2 & & \\ \hline V & & 2 & \end{array}, \begin{array}{c||c|c|c} U & 2 & & \\ \hline V & & 1 & 1 \end{array}, \begin{array}{c||c|c|c} U & 1 & 1 & \\ \hline V & & & 2 \end{array}, \begin{array}{c||c|c|c|c} U & 1 & 1 & & \\ \hline V & & & 1 & 1 \end{array}, \\
\dfrac{(q^2x_i-x_j)(x_i-q^2x_j)}{(x_i-x_j)^2}, & \begin{array}{c||c|c|c} U & & 2 & \\ \hline V & 1 & & 1 \end{array} \begin{array}{c||c|c|c} U & 1 & & 1 \\ \hline V & & 2 & \end{array}, \begin{array}{c||c|c|c|c} U & 1 & & & 1 \\ \hline V & & 1 & 1 & \end{array}, \begin{array}{c||c|c|c|c} U & & 1 & 1 & \\ \hline V & 1 & & & 1 \end{array}, \\
\dfrac{(q^2x_i-x_j)^2(qx_j-x_i)}{(x_i-x_j)^2(qx_i-x_j)}, & \begin{array}{c||c|c|c|c} U & 1 & & 1 & \\ \hline V & & 1 & & 1 \end{array}, \\
\dfrac{(qx_i-x_j)(x_i-q^2x_j)^2}{(qx_j-x_i)(x_i-x_j)^2}, & \begin{array}{c||c|c|c|c} U & & 1 & & 1 \\ \hline V & 1 & & 1 & \end{array}.
\end{cases}$$

$$\Delta_{V,U}(x_i,x_j;z)=\begin{cases}
\dfrac{(q^2x_i-x_j)(x_ix_j+q^2x_ix_j-q^3x_iz_{s_1}-q^3x_jz_{s_1})}{q(-1+q)z_{s_1}(x_i-x_j)(x_j-qx_i)}, & \begin{array}{c||c|c|c} U & 1 & & 1 \\ \hline V & 1 & 1 & \end{array}, \\
\dfrac{(q^2x_j-x_i)(x_ix_j+q^2x_ix_j-q^3x_iz_{s_1}-q^3x_jz_{s_1})}{q(-1+q)z_{s_1}(x_i-x_j)(qx_j-x_i)}, & \begin{array}{c||c|c|c} U & 1 & 1 & \\ \hline V & 1 & & 1 \end{array}, \\
\dfrac{(q^2x_i-x_j)(x_ix_j+q^2x_ix_j-q^2x_iz_{s_2}-q^2x_jz_{s_2})}{(-1+q)(x_i-x_j)(x_j-qx_i)z_{s_2}}, & \begin{array}{c||c|c|c} U & & 1 & 1 \\ \hline V & 1 & & 1 \end{array}, \\
\dfrac{(q^2x_j-x_i)(x_ix_j+q^2x_ix_j-q^2x_iy_{s_2}-q^2x_jy_{s_2})}{(-1+q)(x_i-x_j)(qx_j-x_i)z_{s_2}}, & \begin{array}{c||c|c|c} U & 1 & & 1 \\ \hline V & & 1 & 1 \end{array}, \\
\dfrac{(q^2x_i-x_j)(x_ix_j+q^2x_ix_j-q^3x_iz_{s_2}-q^2x_jz_{s_2})}{q(-1+q)(x_i-x_j)^2z_{s_2}}, & \begin{array}{c||c|c|c} U & & 1 & 1 \\ \hline V & 1 & 1 & \end{array}, \\
\dfrac{(q^2x_j-x_i)(x_ix_j+q^2x_ix_j-q^2x_iz_{s_1}-q^3x_jz_{s_1})}{q(-1+q)(x_i-x_j)^2z_{s_1}}, & \begin{array}{c||c|c|c} U & 1 & 1 & \\ \hline V & & 1 & 1 \end{array}.
\end{cases}$$

$$\Delta_{V,U}(x_i,x_j;z)=\frac{(x_ix_j+q^2x_ix_j-q^3x_iz_{s_1}-q^3x_jz_{s_1})(x_ix_j+q^2x_ix_j-q^2x_iz_{s_2}-q^2x_jz_{s_2})}{q(-1+q)^2(x_j-qx_i)(x_i-qx_j)z_{s_1}z_{s_2}}$$

$$\begin{array}{c||c|c} U & 1 & 1 \\ \hline V & 1 & 1 \end{array}$$

Summary

In this talk, we have discussed

- The rational functions F_S and G_S for the Izergin–Korepin model
- The Cauchy identity for the rational functions F_S and G_S
- The symmetrization formula for the rational functions F_S

Future directions

- Generalize to higher spin (fusion)
- Possibility of "coloured" Izergin–Korepin model [Borodin,Wheeler'18]
- Orthogonality of the rational functions F_S [Borodin,Petrov'16]
- K-matrix for the boundary of the Izergin–Korepin model