Representation Theory of the One-Boundary Temperley-Lieb Algebra

Madeline Nurcombe

The University of Queensland

February 9, 2022

Context

The Temperley-Lieb algebra:

- Temperley & Lieb (1971) — ice-type models, lattice colouring problems

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- Martin (1991), Westbury (1995) — original work

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Representation theory:

- Martin (1991), Westbury (1995) — original work
- Ridout & Saint-Aubin (2014) — review paper

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One-boundary Temperley-Lieb algebra:

- Pearce, Rasmussen & Tipunin (2014) — three-parameter definition

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My goal:

- Adapt the Ridout & Saint-Aubin paper to study the three-parameter one-boundary Temperley-Lieb algebra $1\mathrm{BTL}_n(\beta; \beta_1, \beta_2)$

Diagrams

We now introduce n-diagrams.

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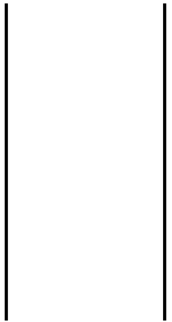
We now introduce *n*-diagrams.

- Two vertical lines



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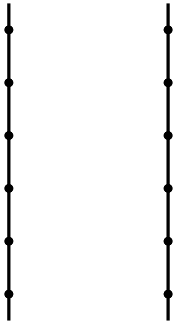
- n nodes on each line (here $n = 6$)



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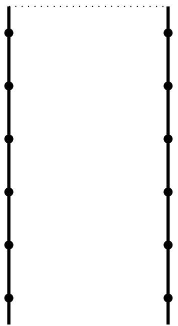

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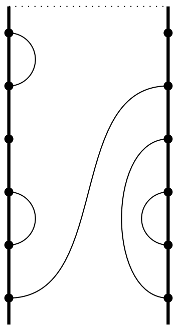

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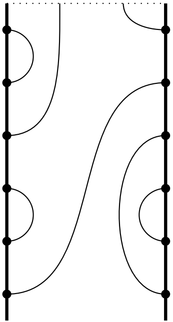

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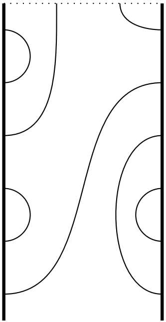

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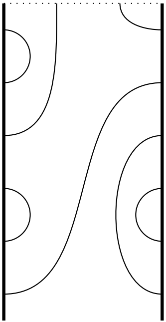

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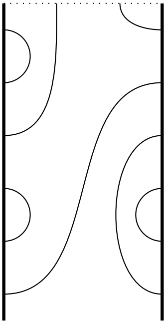

Set of all n-diagrams forms a $\mathbb{C}$-basis for $1\mathrm{BTL}_n$.

Multiplication in $1\mathrm{BTL}_n(\beta; \beta_1, \beta_2)$

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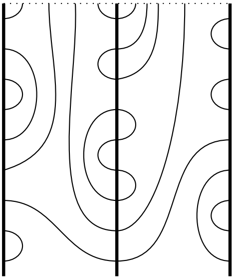

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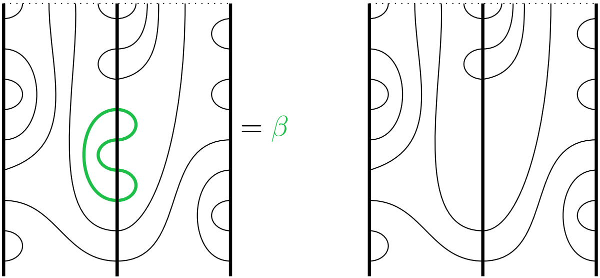

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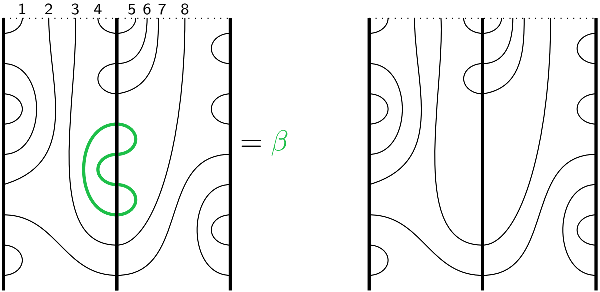

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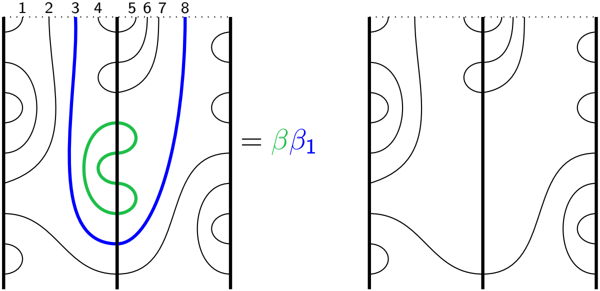

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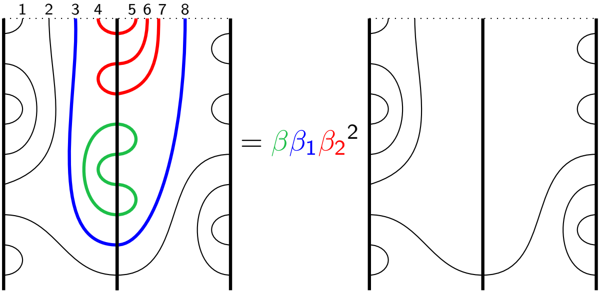

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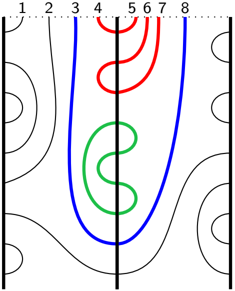
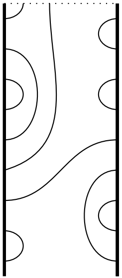


$= \beta\beta_1{\beta_2}^2$



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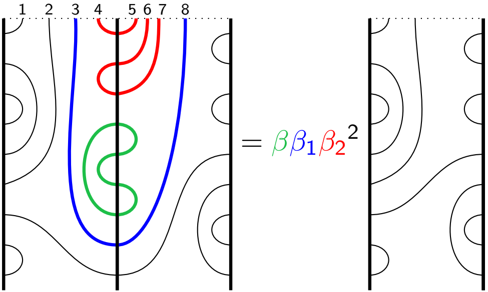

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Standard modules $\mathcal{V}_{n,d}$

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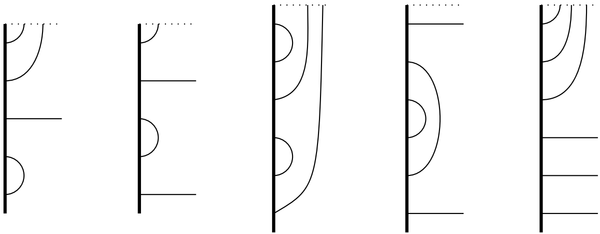

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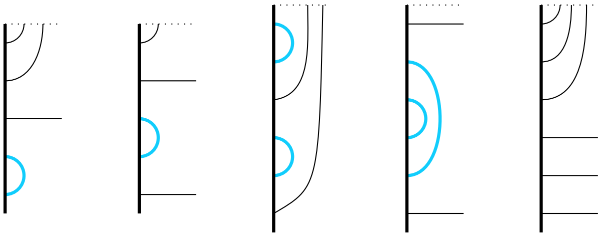

- Nodes can have *links*  ,

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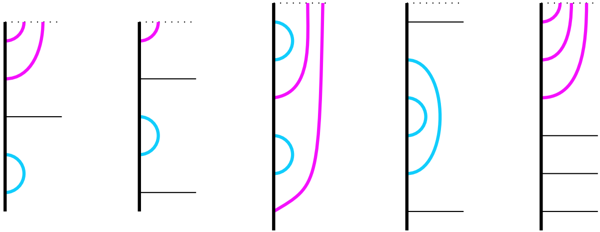
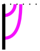


- Nodes can have *links*  , *boundary links*  ,

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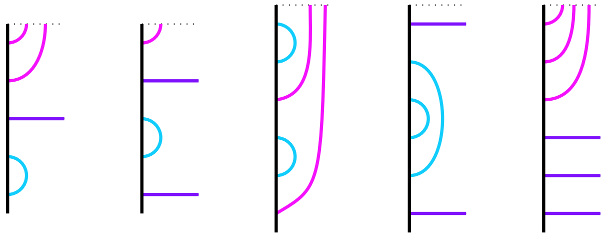
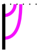
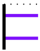


- Nodes can have *links*  , *boundary links*  , *defects* 

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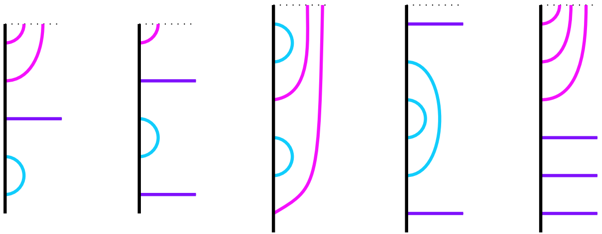
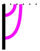
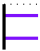


- Nodes can have *links*  , *boundary links*  , *defects* 
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Action on standard modules $\mathcal{V}_{n,d}$

Examples

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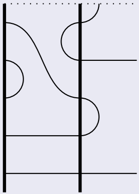
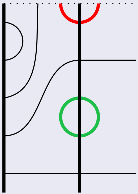
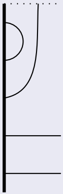
In $\mathcal{V}_{5,2}$, we have



$= 0,$



$= \beta\beta_2$



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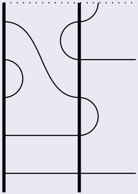
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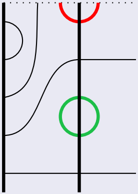
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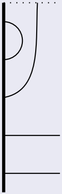
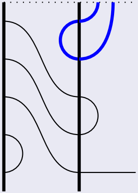
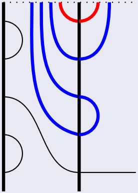

,

while in $\mathcal{V}_{5,1}$ we have


$= \beta_1$

,

$= {\beta_1}^2\beta_2$

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Outer product

From a pair of n-half-diagrams with the same number of defects d, we can construct an n-diagram.

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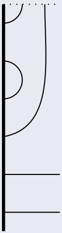
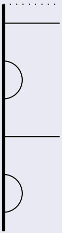
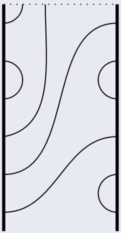
In $\mathcal{V}_{6,2}$, we can take

$x =$  , $y =$  , $|x \;\; y| :=$  $\in 1\mathrm{BTL}_6$.

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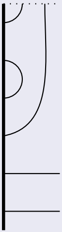
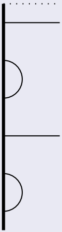
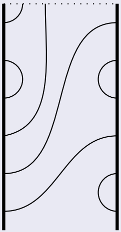
$x =$  , $y =$  , $|x \;\; y| :=$  $\in 1\mathrm{BTL}_6$.

This extends bilinearly to a map $|\cdot \;\; \cdot| : \mathcal{V}_{n,d} \times \mathcal{V}_{n,d} \to 1\mathrm{BTL}_n$.

Bilinear form on $\mathcal{V}_{n,d}$

What if we put two (n, d)-half-diagrams back-to-back instead?

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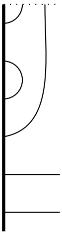
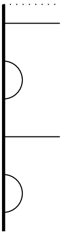
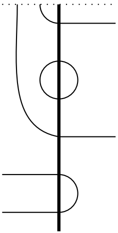
$x =$  , $y =$  , $\langle x, y \rangle =$  $\in \mathbb{C}$

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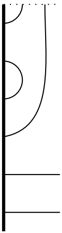
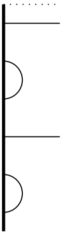
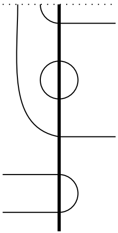
$x =$  , $y =$  , $\langle x, y \rangle =$  $\in \mathbb{C}$

We would like to define such a diagram operation, and extend it bilinearly to $\mathcal{V}_{n,d} \times \mathcal{V}_{n,d}$ to give a bilinear form.

Bilinear form on $\mathcal{V}_{n,d}$

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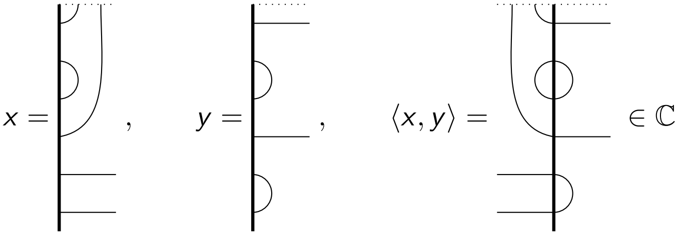

$x =$, $y =$, $\langle x, y \rangle =$ $\in \mathbb{C}$

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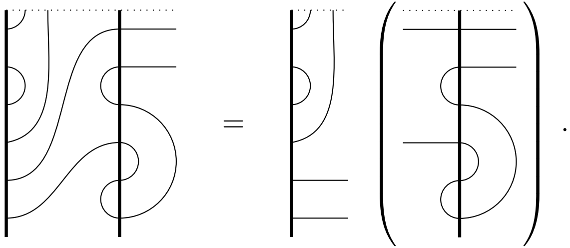
(Not sesquilinear!)

Bilinear form on $\mathcal{V}_{n,d}$

For the bilinear form to be useful, we would like to have

$$|x \quad y|\, z \qquad = \qquad x\,\langle y, z\rangle$$

for all $x, y, z \in \mathcal{V}_{n,d}$, e.g.:



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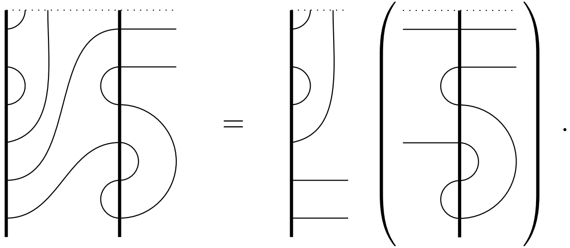
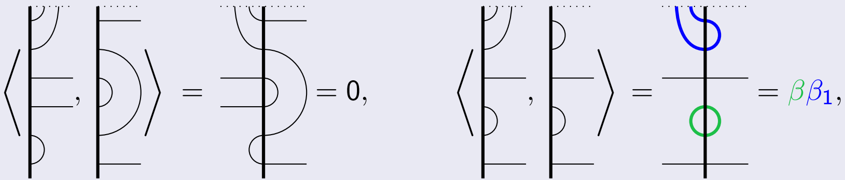


Observing that $|x \quad y|\, z \propto x$ for all $x, y, z \in \mathcal{V}_{n,d}$, this is possible, and uniquely defines a bilinear form $\langle \cdot, \cdot \rangle : \mathcal{V}_{n,d} \times \mathcal{V}_{n,d} \to \mathbb{C}$.

Bilinear form on $\mathcal{V}_{n,d}$

Examples

In $\mathcal{V}_{6,2}$, where $d \equiv n \pmod 2$, we have



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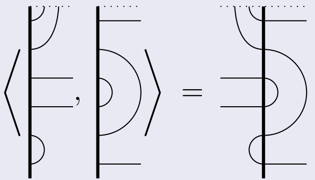
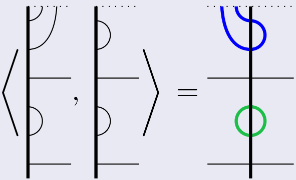
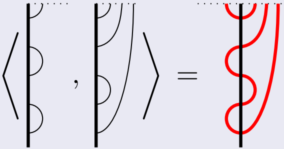


$= 0$,

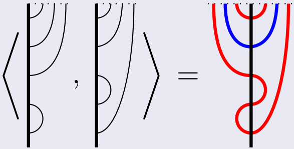


$= \beta\beta_1$,

while in $\mathcal{V}_{5,0}$, where $d \not\equiv n \pmod 2$, we have



$= \beta_2{}^2$,



$= \beta_1\beta_2{}^2$.

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Each of these bilinear forms has an associated *Gram matrix*.

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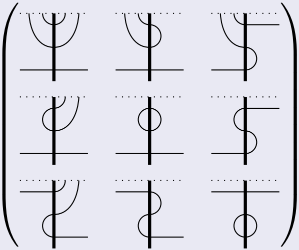

$$= \begin{pmatrix} \beta_1\beta_2 & \beta_1 & 0 \\ \beta_1 & \beta & 1 \\ 0 & 1 & \beta \end{pmatrix}$$

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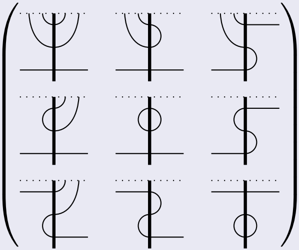

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Note that $\det(G_{n,d})$ is polynomial in β, β_1, β_2.

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Hence we wish to find $\det(G_{n,d})$, and find when $\det(G_{n,d}) = 0$.

Determinant of the Gram matrix

Theorem

For any $\beta, \beta_1, \beta_2 \in \mathbb{C}$, the determinant of the Gram matrix $G_{n,d}$ is given by

$$
\det(G_{n,d}) = \begin{cases}
(-\beta_1)^{\binom{n}{\frac{n-d}{2}-1}} \displaystyle\prod_{j=1}^{\frac{n-d}{2}} \left(\beta_1 U_{d+j-1}\left(\tfrac{\beta}{2}\right) - \beta_2 U_{d+j}\left(\tfrac{\beta}{2}\right)\right)^{\binom{n}{\frac{n-d}{2}-j}} \\
\quad \times \displaystyle\prod_{k=1}^{\frac{n-d}{2}-1} \left(\beta_2 U_{k-1}\left(\tfrac{\beta}{2}\right) - \beta_1 U_k\left(\tfrac{\beta}{2}\right)\right)^{\binom{n}{\frac{n-d}{2}-k-1}}, & d \equiv n \pmod 2, \\
\beta_2^{\binom{n}{\frac{n-d-1}{2}}} \displaystyle\prod_{j=1}^{\frac{n-d-1}{2}} \left(\beta_2 U_{d+j-1}\left(\tfrac{\beta}{2}\right) - \beta_1 U_{d+j}\left(\tfrac{\beta}{2}\right)\right)^{\binom{n}{\frac{n-d-1}{2}-j}} \\
\quad \times \displaystyle\prod_{k=1}^{\frac{n-d-1}{2}} \left(\beta_1 U_{k-1}\left(\tfrac{\beta}{2}\right) - \beta_2 U_k\left(\tfrac{\beta}{2}\right)\right)^{\binom{n}{\frac{n-d-1}{2}-k}}, & d \not\equiv n \pmod 2,
\end{cases}
$$

where U_m is the mth Chebyshev polynomial of the second kind.

When $\det(G_{n,d}) = 0$

Theorem

We have $\det(G_{n,d}) = 0$ *if and only if* $d < n$ *and*

- $\beta_{n,d} = 0$; *or*
- $\beta_{n,d} \neq 0$, $q \neq \pm 1$, *and* $\beta'_{n,d} \notin \{q\beta_{n,d}, q^{-1}\beta_{n,d}\}$, *and*
    - $\xi_{n,d} - (d+j+1)\lambda \in \pi\mathbb{Z}$ *for some* $j \in \mathbb{Z}$ *with* $1 \leq j \leq \left\lfloor \frac{n-d}{2} \right\rfloor$, *or*
    - $\xi_{n,d} + k\lambda \in \pi\mathbb{Z}$ *for some* $k \in \mathbb{Z}$ *with* $1 \leq k \leq \left\lfloor \frac{n-d-1}{2} \right\rfloor$; *or*
- $\beta_{n,d} \neq 0$, $q = \pm 1$, *and*
    - $\beta'_{n,d} = \frac{d+j}{d+j+1} q^{-1}\beta_{n,d}$ *for some* $j \in \mathbb{Z}$ *with* $1 \leq j \leq \left\lfloor \frac{n-d}{2} \right\rfloor$, *or*
    - $\beta'_{n,d} = \frac{k+1}{k} q\,\beta_{n,d}$ *for some* $k \in \mathbb{Z}$ *with* $1 \leq k \leq \left\lfloor \frac{n-d-1}{2} \right\rfloor$,

where $\beta = q + q^{-1}$, $q = e^{i\lambda}$, *and* $\xi_{n,d}$ *comes from the parametrisation*

$$\beta'_{n,d} = \frac{q - q^{-1}e^{2i\xi_{n,d}}}{1 - e^{2i\xi_{n,d}}}\beta_{n,d},$$

where $\beta_{n,d} = \beta_1$ *and* $\beta'_{n,d} = \beta_2$ *if* $d \equiv n \mod 2$, *or vice versa if* $d \not\equiv n \mod 2$.

Cellularity of $\mathrm{1BTL}_n(\beta; \beta_1, \beta_2)$

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- $1\mathrm{BTL}_n(\beta;\beta_1,\beta_2)$ is semisimple if and only if $\det(G_{n,d}) \neq 0$ for all $d = 0,1,\ldots,n$. Hence $1\mathrm{BTL}_n(\beta;\beta_1,\beta_2)$ is semisimple for generic parameter values.

Composition series

Let A be a finite-dimensional algebra, and V a finite-dimensional A-module.

A *composition series* of V is a sequence of submodules

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By the Jordan-Hölder theorem, V has a unique multiset of composition factors.

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Loewy diagram of A-module V:

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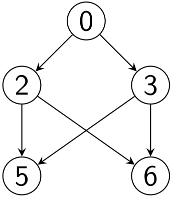

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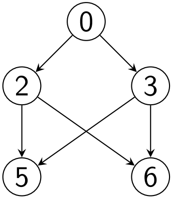

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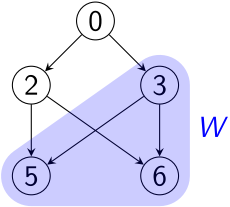

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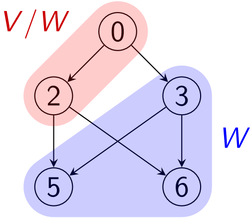

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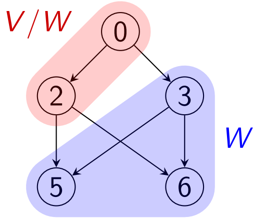

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- Well-defined if, e.g., V has no repeated composition factors, or V has a unique composition series

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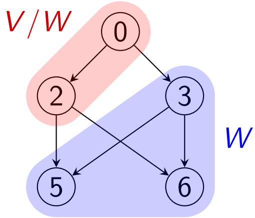

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Examples

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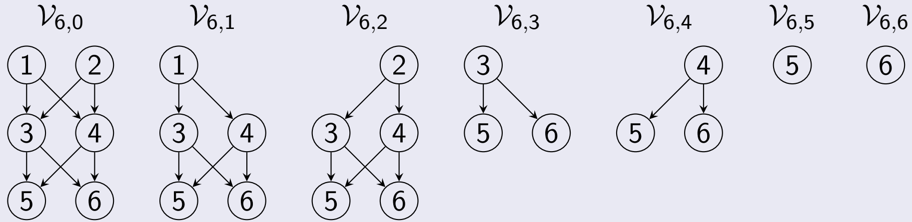

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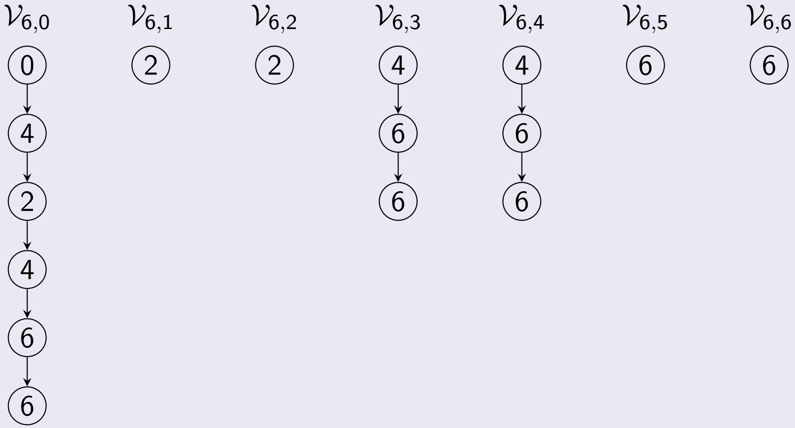

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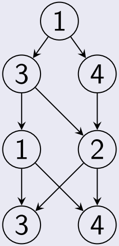
- Consider the Hamiltonians for $1\mathrm{BTL}_n$ lattice models, and their eigenvalues, as well as the continuum scaling limit $n \to \infty$
- Construct and study a one-boundary version of the BMW algebra

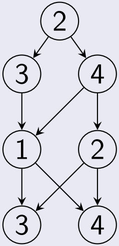
Principal indecomposable and induced modules

Principal indecomposable modules with $(\beta, \beta_1, \beta_2) = (0, 0, 1)$

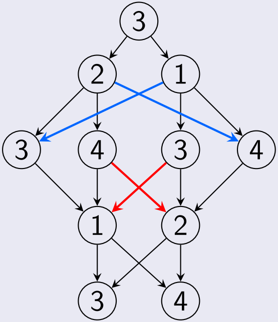
$\mathcal{P}_{4,1}$:



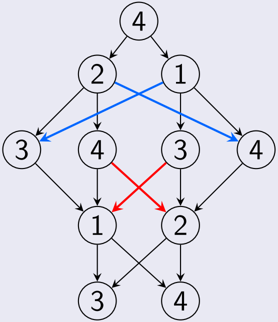
$\mathcal{P}_{4,2}$:



$\mathcal{P}_{4,3}$:



$\mathcal{P}_{4,4}$:

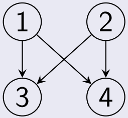


Principal indecomposable and induced modules

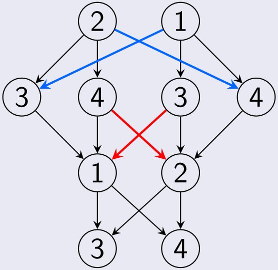
Define $\mathrm{Ind}_B^A V := A \otimes_B V$.

Induced modules with $(\beta, \beta_1, \beta_2) = (0, 0, 1)$

$\mathrm{Ind}_{\mathrm{TL}_4}^{1\mathrm{BTL}_4}(\mathcal{TLV}_{4,0})$:



$\mathrm{Ind}_{\mathrm{TL}_4}^{1\mathrm{BTL}_4}(\mathcal{TLV}_{4,2})$:



$\mathrm{Ind}_{\mathrm{TL}_4}^{1\mathrm{BTL}_4}(\mathcal{TLV}_{4,4})$: