

Representation Theory of the One-Boundary Temperley-Lieb Algebra

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The Temperley-Lieb algebra:

- Temperley & Lieb (1971) — ice-type models, lattice colouring problems

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Representation theory:

- Martin (1991), Westbury (1995) — original work
- Ridout & Saint-Aubin (2014) — review paper

One-boundary Temperley-Lieb algebra:

- Pearce, Rasmussen & Tipunin (2014) — three-parameter definition

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My goal:

- Adapt the Ridout & Saint-Aubin paper to study the three-parameter one-boundary Temperley-Lieb algebra $1\text{BTL}_n(\beta; \beta_1, \beta_2)$

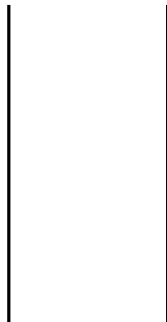
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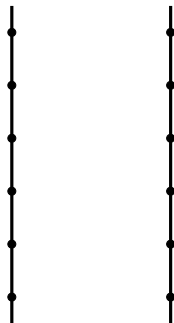
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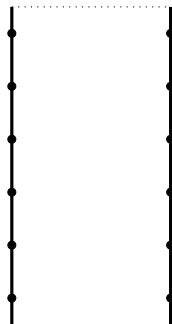
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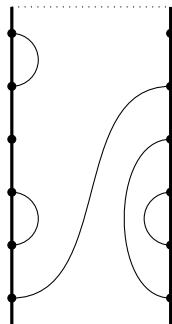
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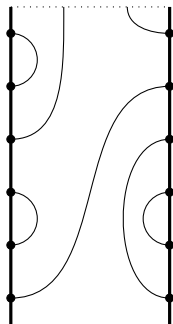
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- Strings may connect pairs of nodes



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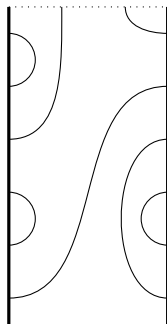
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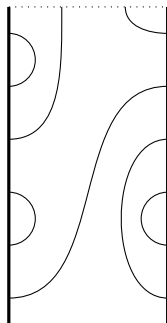
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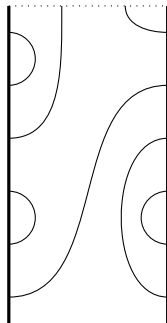
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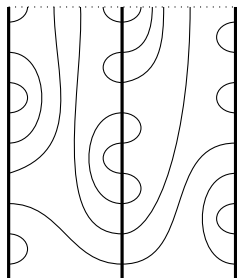
Set of all n -diagrams forms a \mathbb{C} -basis for 1BTL_n .

Multiplication in $1\text{BTL}_n(\beta; \beta_1, \beta_2)$

We now want to multiply n -diagrams.

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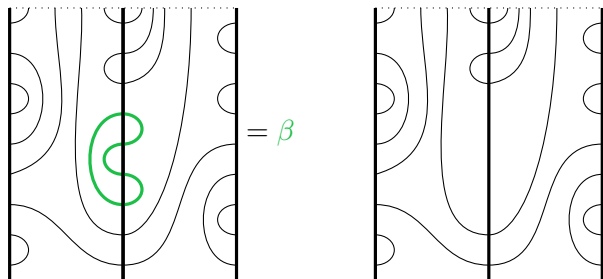
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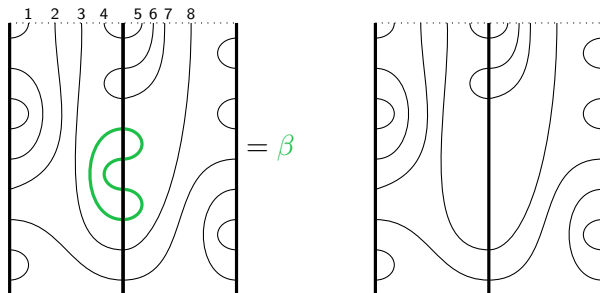
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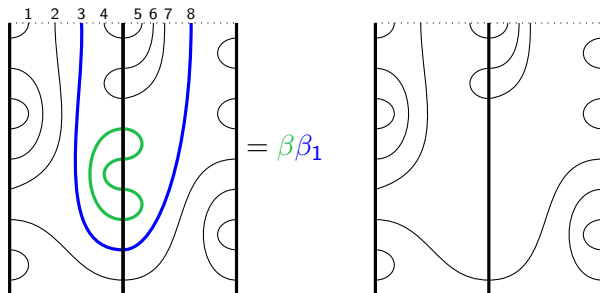
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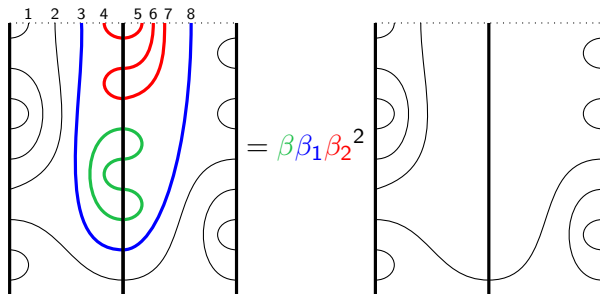
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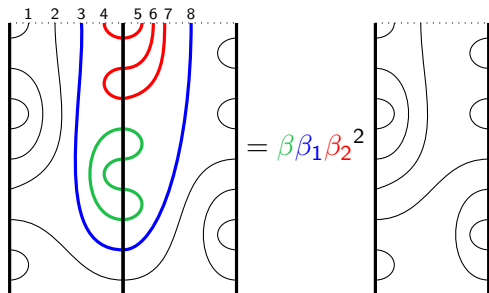
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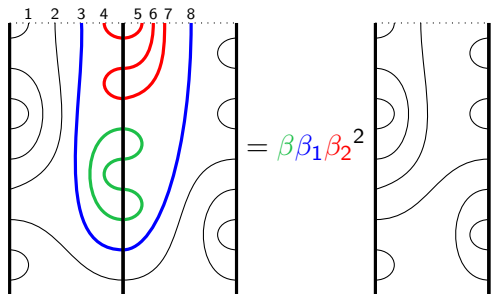
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- Remove middle line, tighten strings
- Extend bilinearly to $1\text{BTL}_n(\beta; \beta_1, \beta_2)$

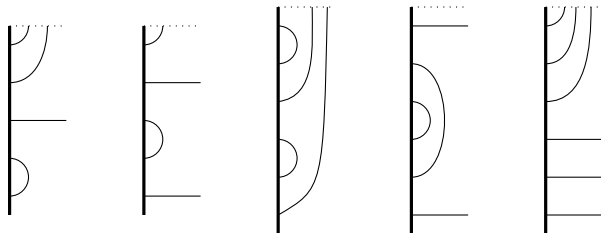
Standard modules $\mathcal{V}_{n,d}$

We now want to construct a representation of $1\text{BTL}_n(\beta; \beta_1, \beta_2)$.

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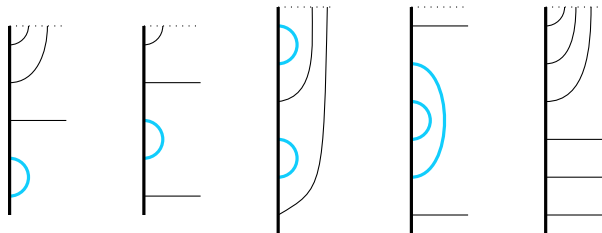
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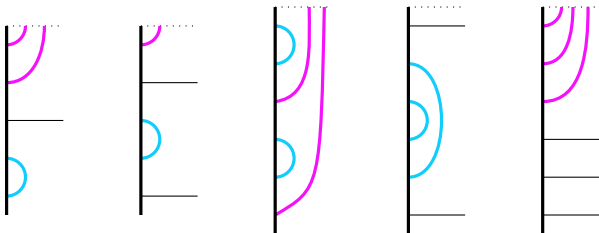


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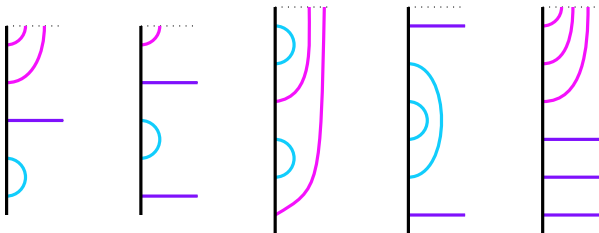





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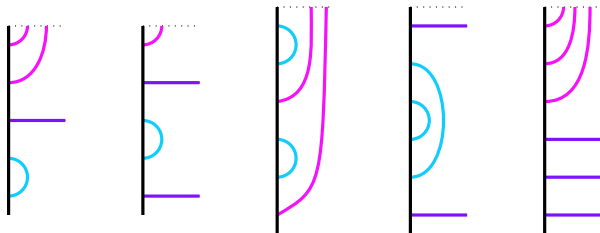





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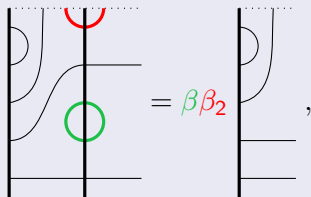
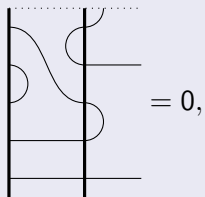


- Nodes can have *links* , *boundary links* , *defects* 
- Set of all n -half diagrams with d defects forms a \mathbb{C} -basis for *standard module* $\mathcal{V}_{n,d}$

Examples

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In $\mathcal{V}_{5,2}$, we have



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In $\mathcal{V}_{5,2}$, we have

$$= 0,$$

$$= \beta_1 \beta_2,$$

while in $\mathcal{V}_{5,1}$ we have

$$= \beta_1,$$

$$= \beta_1^2 \beta_2.$$

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Example

In $\mathcal{V}_{6,2}$, we can take

The diagrammatic equation shows the outer product of two half-diagrams. On the left, $x =$ is a half-diagram with a vertical line on the left and a curved line on the right that starts at the top and ends at the bottom. In the middle, $y =$ is a half-diagram with a vertical line on the left and a horizontal line on the right. On the right, $|x \ y| :=$ is the resulting n -diagram, which is the composition of x and y . To the right of this diagram is the expression $\in 1\text{BTL}_6$.

$$x = \text{[diagram]}, \quad y = \text{[diagram]}, \quad |x \ y| := \text{[diagram]} \in 1\text{BTL}_6.$$

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$$x = \begin{array}{|c|} \hline \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \hline \end{array}, \quad y = \begin{array}{|c|} \hline \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \hline \end{array}, \quad |x \ y| := \begin{array}{|c|} \hline \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \hline \end{array} \in 1\text{BTL}_6.$$

This extends bilinearly to a map $|\cdot \ \cdot| : \mathcal{V}_{n,d} \times \mathcal{V}_{n,d} \rightarrow 1\text{BTL}_n$.

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What if we put two (n, d) -half-diagrams back-to-back instead?

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$$x = \text{diagram 1}, \quad y = \text{diagram 2}, \quad \langle x, y \rangle = \text{diagram 3} \in \mathbb{C}$$

The diagram for x consists of a vertical line with two horizontal lines extending to the right from its lower part. From the upper part of the vertical line, a curve starts, goes up and right, then curves back down and left to meet the vertical line, forming a loop. A dotted horizontal line is drawn above the top of this loop.

The diagram for y consists of a vertical line with two horizontal lines extending to the right from its upper part. From the lower part of the vertical line, a semi-circle extends to the right.

The diagram for $\langle x, y \rangle$ consists of a vertical line with two horizontal lines extending to the right from its lower part. From the upper part of the vertical line, a curve starts, goes up and right, then curves back down and left to meet the vertical line, forming a loop. A circle is drawn inside this loop, centered on the vertical line. A dotted horizontal line is drawn above the top of this loop.

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What if we put two (n, d) -half-diagrams back-to-back instead?

The diagram shows three half-diagrams separated by commas. The first is labeled $x =$ and consists of a vertical line with two horizontal lines at the bottom. A semi-circle on the right side of the line connects the top of the line to the top of the upper horizontal line. A second semi-circle on the right side connects the top of the line to the top of the lower horizontal line. The second diagram is labeled $y =$ and consists of a vertical line with two horizontal lines at the bottom. A semi-circle on the left side of the line connects the top of the line to the top of the upper horizontal line. A second semi-circle on the left side connects the top of the line to the top of the lower horizontal line. The third diagram is labeled $\langle x, y \rangle =$ and consists of a vertical line with two horizontal lines at the bottom. A circle is drawn around the vertical line, centered between the two horizontal lines. The diagram is followed by $\in \mathbb{C}$.

We would like to define such a diagram operation, and extend it bilinearly to $\mathcal{V}_{n,d} \times \mathcal{V}_{n,d}$ to give a bilinear form.

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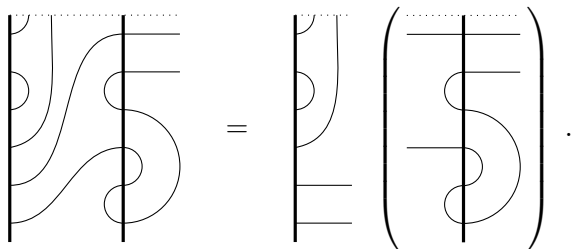
(Not sesquilinear!)

Bilinear form on $\mathcal{V}_{n,d}$

For the bilinear form to be useful, we would like to have

$$|x \ y|z = x \langle y, z \rangle$$

for all $x, y, z \in \mathcal{V}_{n,d}$, e.g.:

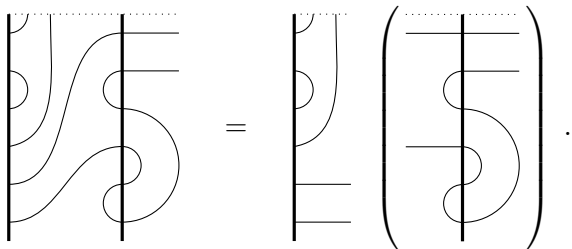


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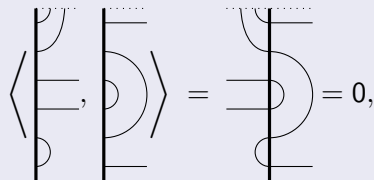
The diagram shows an equality between two expressions. On the left is a single complex diagram with two vertical lines and several curved lines connecting them. On the right is the product of two diagrams: the first is simpler, and the second is enclosed in large parentheses.

Observing that $|x \ y|z \propto x$ for all $x, y, z \in \mathcal{V}_{n,d}$, this is possible, and uniquely defines a bilinear form $\langle \cdot, \cdot \rangle : \mathcal{V}_{n,d} \times \mathcal{V}_{n,d} \rightarrow \mathbb{C}$.

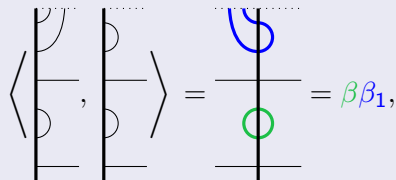
Bilinear form on $\mathcal{V}_{n,d}$

Examples

In $\mathcal{V}_{6,2}$, where $d \equiv n \pmod{2}$, we have



A diagrammatic equation showing the evaluation of a bilinear form. On the left, two diagrams are separated by a comma. The first diagram has a vertical line with four horizontal strands. The top two strands are connected by a cap, and the bottom two strands are connected by a cup. The second diagram has a vertical line with four horizontal strands. The top two strands are connected by a cap, and the bottom two strands are connected by a cup. The two diagrams are enclosed in large angle brackets. This is followed by an equals sign and a single diagram with a vertical line and four horizontal strands. The top two strands are connected by a cap, and the bottom two strands are connected by a cup. This diagram is also enclosed in large angle brackets. The entire expression is followed by an equals sign and the number 0.



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Bilinear form on $\mathcal{V}_{n,d}$

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In $\mathcal{V}_{6,2}$, where $d \equiv n \pmod{2}$, we have

$$\langle \text{crossing}, \text{crossing} \rangle = \text{crossing} = 0, \quad \langle \text{crossing}, \text{crossing} \rangle = \text{blue loop} + \text{green circle} = \beta\beta_1,$$

while in $\mathcal{V}_{5,0}$, where $d \not\equiv n \pmod{2}$, we have

$$\langle \text{crossing}, \text{crossing} \rangle = \text{red loop} = \beta_2^2, \quad \langle \text{crossing}, \text{crossing} \rangle = \text{blue loop} + \text{red loop} = \beta_1\beta_2^2.$$

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Each of these bilinear forms has an associated *Gram matrix*.

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$$G_{3,1} = \begin{pmatrix} \begin{array}{c} \text{---} \\ \text{U} \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ \text{S} \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ \text{S} \\ \text{---} \end{array} \\ \begin{array}{c} \text{---} \\ \text{U} \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ \text{O} \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ \text{S} \\ \text{---} \end{array} \\ \begin{array}{c} \text{---} \\ \text{U} \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ \text{S} \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ \text{O} \\ \text{---} \end{array} \end{pmatrix} = \begin{pmatrix} \beta_1\beta_2 & \beta_1 & 0 \\ \beta_1 & \beta & 1 \\ 0 & 1 & \beta \end{pmatrix}$$

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Note that $\det(G_{n,d})$ is polynomial in β, β_1, β_2 .

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- If $G_{n,d} \neq 0$, then the property $|x \ y| z = x \langle y, z \rangle$ implies

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- If $G_{n,d} = 0$,
 - $d \equiv n \pmod{2}$: $\mathcal{V}_{n,d}$ is reducible
 - $d \not\equiv n \pmod{2}$: $\mathcal{V}_{n,d}$ is irreducible $\Leftrightarrow \det(G_{n,d+1}) \neq 0$.

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 - $d \not\equiv n \pmod{2}$: $\mathcal{V}_{n,d}$ is irreducible $\Leftrightarrow \det(G_{n,d+1}) \neq 0$.

Hence we wish to find $\det(G_{n,d})$, and find when $\det(G_{n,d}) = 0$.

Determinant of the Gram matrix

Theorem

For any $\beta, \beta_1, \beta_2 \in \mathbb{C}$, the determinant of the Gram matrix $G_{n,d}$ is given by

$$\det(G_{n,d}) = \begin{cases} (-\beta_1)^{\binom{\frac{n-d}{2}-1}{\frac{n-d}{2}}} \prod_{j=1}^{\frac{n-d}{2}} \left(\beta_1 U_{d+j-1} \left(\frac{\beta}{2} \right) - \beta_2 U_{d+j} \left(\frac{\beta}{2} \right) \right)^{\binom{\frac{n-d}{2}-j}{\frac{n-d}{2}}} \\ \quad \times \prod_{k=1}^{\frac{n-d}{2}-1} \left(\beta_2 U_{k-1} \left(\frac{\beta}{2} \right) - \beta_1 U_k \left(\frac{\beta}{2} \right) \right)^{\binom{\frac{n-d}{2}-k-1}{\frac{n-d}{2}}}, & d \equiv n \pmod{2}, \\ \beta_2^{\binom{\frac{n-d-1}{2}}{\frac{n-d-1}{2}}} \prod_{j=1}^{\frac{n-d-1}{2}} \left(\beta_2 U_{d+j-1} \left(\frac{\beta}{2} \right) - \beta_1 U_{d+j} \left(\frac{\beta}{2} \right) \right)^{\binom{\frac{n-d-1}{2}-j}{\frac{n-d-1}{2}}} \\ \quad \times \prod_{k=1}^{\frac{n-d-1}{2}} \left(\beta_1 U_{k-1} \left(\frac{\beta}{2} \right) - \beta_2 U_k \left(\frac{\beta}{2} \right) \right)^{\binom{\frac{n-d-1}{2}-k}{\frac{n-d-1}{2}}}, & d \not\equiv n \pmod{2}, \end{cases}$$

where U_m is the m th Chebyshev polynomial of the second kind.

When $\det(G_{n,d}) = 0$

Theorem

We have $\det(G_{n,d}) = 0$ if and only if $d < n$ and

- $\beta_{n,d} = 0$; or
- $\beta_{n,d} \neq 0$, $q \neq \pm 1$, and $\beta'_{n,d} \notin \{q\beta_{n,d}, q^{-1}\beta_{n,d}\}$, and
 - $\xi_{n,d} - (d+j+1)\lambda \in \pi\mathbb{Z}$ for some $j \in \mathbb{Z}$ with $1 \leq j \leq \lfloor \frac{n-d}{2} \rfloor$, or
 - $\xi_{n,d} + k\lambda \in \pi\mathbb{Z}$ for some $k \in \mathbb{Z}$ with $1 \leq k \leq \lfloor \frac{n-d-1}{2} \rfloor$; or
- $\beta_{n,d} \neq 0$, $q = \pm 1$, and
 - $\beta'_{n,d} = \frac{d+j}{d+j+1} q^{-1} \beta_{n,d}$ for some $j \in \mathbb{Z}$ with $1 \leq j \leq \lfloor \frac{n-d}{2} \rfloor$, or
 - $\beta'_{n,d} = \frac{k+1}{k} q \beta_{n,d}$ for some $k \in \mathbb{Z}$ with $1 \leq k \leq \lfloor \frac{n-d-1}{2} \rfloor$,

where $\beta = q + q^{-1}$, $q = e^{i\lambda}$, and $\xi_{n,d}$ comes from the parametrisation

$$\beta'_{n,d} = \frac{q - q^{-1} e^{2i\xi_{n,d}}}{1 - e^{2i\xi_{n,d}}} \beta_{n,d},$$

where $\beta_{n,d} = \beta_1$ and $\beta'_{n,d} = \beta_2$ if $d \equiv n \pmod{2}$, or vice versa if $d \not\equiv n \pmod{2}$.

Cellularity of $1\text{BTL}_n(\beta; \beta_1, \beta_2)$

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- A complete set of non-isomorphic finite-dimensional irreducible modules is given by

$$\left\{ \mathcal{V}_{n,d} / \mathcal{R}_{n,d} \mid \langle \cdot, \cdot \rangle_{n,d} \not\equiv 0, d \in \{0, 1, \dots, n\} \right\},$$

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- $1\text{BTL}_n(\beta; \beta_1, \beta_2)$ is semisimple if and only if $\det(G_{n,d}) \neq 0$ for all $d = 0, 1, \dots, n$. Hence $1\text{BTL}_n(\beta; \beta_1, \beta_2)$ is semisimple for generic parameter values.

Composition series

Let A be a finite-dimensional algebra, and V a finite-dimensional A -module.

A *composition series* of V is a sequence of submodules

$$0 = V_0 \subset V_1 \subset \cdots \subset V_m = V$$

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By the Jordan-Hölder theorem, V has a unique multiset of composition factors.

Loewy diagrams

We write $\textcircled{d} = \mathcal{V}_{n,d}/\mathcal{R}_{n,d}$.

Loewy diagrams

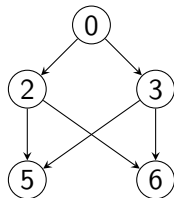
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Loewy diagram of A -module V :

- Directed graph

$$(\beta, \beta_1, \beta_2) = (1, 1, 1)$$

$$V = \mathcal{V}_{6,0} :$$



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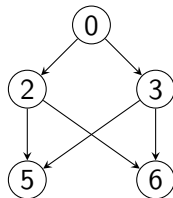
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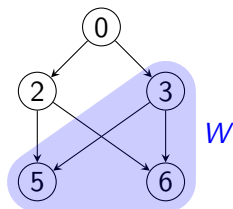
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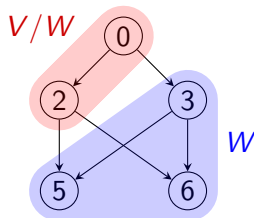
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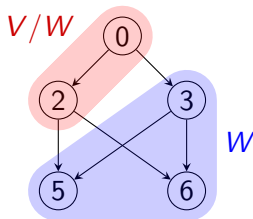
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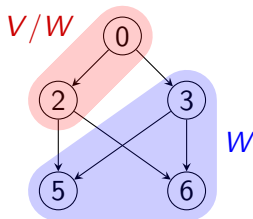
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- Well-defined if, e.g., V has no repeated composition factors, or V has a unique composition series

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Standard module Loewy diagrams

- We have conjectured the Loewy diagram for each standard module for all parameter values, and have proven some of them

Standard module Loewy diagrams

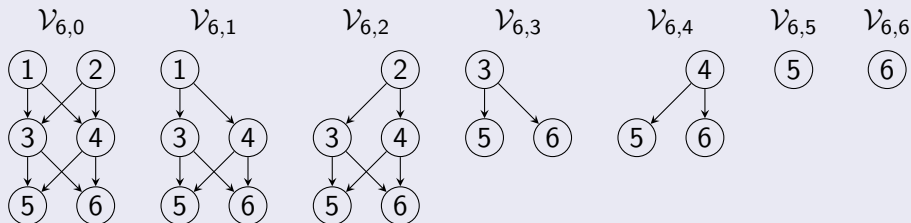
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Examples

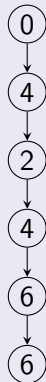
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Standard module Loewy diagrams

Examples

$$(\beta, \beta_1, \beta_2) = (1, 0, 0)$$

 $\mathcal{V}_{6,0}$  $\mathcal{V}_{6,1}$  $\mathcal{V}_{6,2}$  $\mathcal{V}_{6,3}$  $\mathcal{V}_{6,4}$  $\mathcal{V}_{6,5}$  $\mathcal{V}_{6,6}$ 

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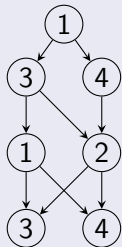
Branching out:

- Consider the Hamiltonians for $1\mathrm{BTL}_n$ lattice models, and their eigenvalues, as well as the continuum scaling limit $n \rightarrow \infty$
- Construct and study a one-boundary version of the BMW algebra

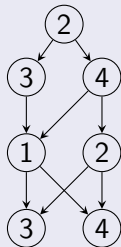
Principal indecomposable and induced modules

Principal indecomposable modules with $(\beta, \beta_1, \beta_2) = (0, 0, 1)$

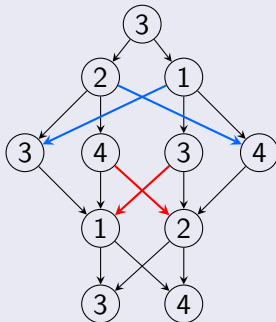
$\mathcal{P}_{4,1}$:



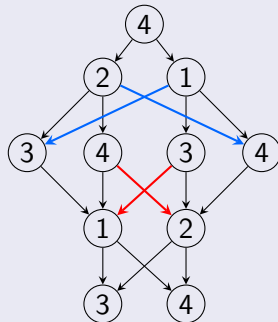
$\mathcal{P}_{4,2}$:



$\mathcal{P}_{4,3}$:



$\mathcal{P}_{4,4}$:

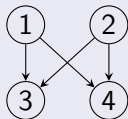


Principal indecomposable and induced modules

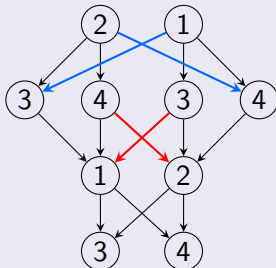
Define $\text{Ind}_B^A V := A \otimes_B V$.

Induced modules with $(\beta, \beta_1, \beta_2) = (0, 0, 1)$

$\text{Ind}_{\text{TL}_4}^{1\text{BTL}_4}(\mathcal{TLV}_{4,0}) :$



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