W-Algebras Related to $\mathfrak{sl}_3$

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Non-rational CFTs

A conformal field theory (CFT) is a quantum field theory with conformal symmetry. Axiomatization of the 'chiral symmetry algebra' of a 2D CFT is given by vertex operator algebras [Frenkel, Lepowsky, Meurman, '88].

Representation Theoretic Data for a 2D CFT

- A vertex operator algebra V,
- A category $\mathcal{C}$ of V-modules satisfying a long list of conditions.

- A CFT is rational if $\mathcal{C}$ is semisimple and has finitely many simple V-modules, many nice examples and general results.
- 'Building blocks' are the WZW models at positive-integer level $L_k(\mathfrak{g})$ ($\mathcal{C}$ a highest-weight category).
- New applications require non-rational (or logarithmic) CFTs, few examples and poorly understood in general.
- 'Building blocks' may be WZW models at fractional level? What should $\mathcal{C}$ be? $\rightarrow$ relaxed highest-weight modules + spectral flow?

Vertex Operator Algebras

Definition

A vertex operator algebra (VOA) consists of a $\mathbb{C}$-graded vector space V and to each $a \in V_h$, a field

$$a(z) = \sum_{n \in \mathbb{Z}-h} a_n z^{-n-h}$$

where $a_n \in End(V)$ are the modes of a. Moreover, there is an element $L \in V_2$ whose modes satisfy the Virasoro commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

and $L_0|_{V_h} = h\mathrm{Id}_{V_h}$. The field $L(z)$ is called the energy-momentum field.

- Subject to a long list of axioms, both algebraic and analytic (e.g. how does the product $a(z)b(w)$ relate to the product $b(z)a(w)$).
- Examples are $L_k(\mathfrak{g})$ and Vir_c (same notation for a CFT and its VOA).

VOA Modules

Definition

Let V be a VOA. A V-module is a vector space M with an action of the fields $a(z)$ (and therefore of the modes a_n) of V satisfying:

- M is graded by L_0 eigenvalues,
- Some conditions.

If the L_0 grading is bounded below, M is a positive-energy module.

Also have twisted modules if certain V_h are non-zero ($h \in \mathbb{Z} + \frac{1}{2}$).

Goal

Understand the category of relaxed h.w. modules + spectral flows for fractional-level WZW models, starting with $L_k(\mathfrak{sl}_3)$ at admissible levels.

- Have a lot of information about $L_k(\mathfrak{sl}_2)$ at admissible levels [Creutzig-Ridout '15].
- Characters of relaxed h.w. modules which aren't h.w. modules involve characters of a Virasoro minimal model (related to $\mathfrak{sl}_2$ by quantum Hamiltonian reduction).

Lesson

Studying the relaxed h.w. modules for $L_k(\mathfrak{g})$ at fractional-levels will likely require knowledge of the VOAs related to $\mathfrak{g}$ by quantum Hamiltonian reduction.

Quantum Hamiltonian Reduction

First examples from physics and later made systematic [Feigen, Frenkel '90].

$$\left\{ \begin{array}{c} \text{Simple, fin-dim Lie superalgebra } \mathfrak{g}, \\ \text{an embedding } \mathfrak{sl}_2 \hookrightarrow \mathfrak{g}, \\ k \in \mathbb{C}. \end{array} \right\} \xrightarrow{qHR} W^k(\mathfrak{g}, x, f)$$

This new VOA is called a W-algebra [Kac, Roan, Wakimoto '03].

Example: $\mathfrak{g} = \mathfrak{sl}_2, k + 2 = \frac{u}{v}$ where $u, v \in \mathbb{Z}_{\geq 2}$ are coprime

Two non-isomorphic W-algebras:

- The universal WZW VOA $V^k(\mathfrak{sl}_2)$. Simple quotient is the simple WZW VOA $L_k(\mathfrak{sl}_2)$ (non-rational).
- The universal Virasoro VOA Vir_c for a certain value for c. Simple quotient is the famous Virasoro minimal model $Vir(u, v)$ (rational).

A Higher Rank Example

Example: $\mathfrak{g} = \mathfrak{sl}_3, k + 3 = \frac{u}{v}$ where $u, v \in \mathbb{Z}_{\geq 3}$ are coprime

Three non-isomorphic W-algebras:

- The universal WZW VOA $V^k(\mathfrak{sl}_3)$. Simple quotient is the simple WZW VOA $L_k(\mathfrak{sl}_3)$ (non-rational).
- The universal principal W-algebra W_3^k. Simple quotient is the famous W_3 minimal model $W_3(u, v)$ (rational).
- The universal Bershadsky-Polyakov algebra BP^k. Simple quotient is the simple Bershadsky-Polyakov algebra BP_k (??)

Properties

- BP^k is generated by 4 fields: $J(z), L(z), G^{\pm}(z)$ with L_0 eigenvalue $1, 2, \frac{3}{2}$ respectively and a (known) set of OPEs.
- $BP_k = BP^k / J$ for some VOA ideal J.

Sub-Goal

Classify simple relaxed h.w. modules for BP_k for $k+3=\frac{u}{v}$ where $u, v+1 \in \mathbb{Z}_{\geq 3}$ and $(u,v)=1$ (twisted and untwisted).

Main Tool: Zhu Technology

For any VOA V, there is a corresponding unital, associative algebra $\mathsf{Zhu}[V]$ [Zhu, '90]. This is the (untwisted) Zhu's algebra.

$\mathsf{Zhu}[V]$ is the algebra of zero modes a_0, where $a(z)$ is a field in V, acting on the top space M_{top} of positive-energy modules M.

Theorem [Zhu, '90]

There is a bijective correspondence between isomorphism classes of simple positive-energy V-modules and simple $\mathsf{Zhu}[V]$-modules.

Replacing 'modules' with 'twisted modules' in the above gives the twisted Zhu's algebra $\mathsf{Zhu}^{\tau}[V]$, the theorem above still holds [Dong-Li-Mason, '95].

Zhu with Universal Bershadsky-Polyakov Algebras

Zhu-ified Sub-Goal

Classify simple 'weight' modules for $\mathrm{Zhu}[BP_k]$ and $\mathrm{Zhu}^{\tau}[BP_k]$ with finite-dimensional weight spaces.

First step: What are $\mathrm{Zhu}\left[BP^k\right]$ and $\mathrm{Zhu}^{\tau}\left[BP^k\right]$?

Result

$$\mathrm{Zhu}\left[BP^k\right] \simeq \mathbb{C}[J, L]$$

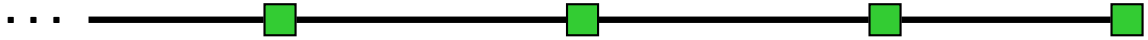
$$\mathrm{Zhu}^{\tau}\left[BP^k\right] \simeq \langle J, G^{\pm}, L | L \text{ central }, [J, G^{\pm}] = \pm G^{\pm}, [G^+, G^-] = f(J, L)\rangle$$

where $f(J, L)$ is a known polynomial [Arakawa '13].

Weight modules for the twisted Zhu's algebra

Finite-dimensional highest-weight/lowest weight:

Infinite-dimensional highest-weight:



Infinite-dimensional lowest-weight:

Infinite-dimensional, not highest-weight or lowest-weight ('dense'):

Result

The precise structure of these modules with help from [Smith, '90]

From Universal to Simple

- By Zhu, this is equivalent to a classification of simple relaxed highest-weight BP^k modules (twisted and untwisted).
- **Question:** Which of these are also modules for $BP_k = BP^k / J$?
- **Answer:** Those annihilated by J.

Problem

The ideal J is not known explicitly.

Workaround

- Use results from [Arakawa, '05,'14], qHR and spectral flow to find all simple h.w. BP^k modules annihilated by J (twisted and untwisted).
- A notion of coherent families [Mathieu, '00] gets us the rest of the simple relaxed h.w. twisted BP_k modules

Results

Let k be such that $k+3=\frac{u}{v}$ where $u, v+1 \in \mathbb{Z}_{\geq 3}$ and $(u,v)=1$.

Theorem [ZF-Kawasetsu-Ridout]

Classification of simple untwisted and twisted relaxed highest-weight modules for BP_k with finite-dimensional weight spaces.

- Set of modules is closed under a conjugation automorphism, required for CFT.
- Parametrised in terms of $\mathfrak{sl}_3$ weights.
- When $v=2$, finitely-many simple modules $\rightarrow$ rational!
- When $v \geq 3$ infinitely-many simple modules $\rightarrow$ non-rational!
- Non-rational VOA where ‘CFT data’ is within reach.
- In the $v \geq 3$/non-rational case, have 1-parameter families of simple dense twisted BP_k modules (relaxed h.w. but not h.w.).

Relating Reductions

Recall: the Virasoro minimal model $Vir(u, v)$ gave us a lot of information about non-h.w. modules for $L_k(\mathfrak{sl}_2)$ where $k + 2 = \frac{u}{v}$.

Question

Does the W_3 minimal model $W_3(u, v)$ tell us anything about non-h.w. twisted modules for BP_k at $k + 3 = \frac{u}{v}$ for $v \geq 3$?

Answer

Yes, in almost the exact same way as the $\mathfrak{sl}_2$ case (details to come).

Future Directions

- Indecomposable BP_k-modules.
- $\mathcal{C}$ is not semisimple, projective covers?.
- Characters and fusion rules, modular transformations, logarithmic version of the Verlinde formula?
- Gain some insight into $L_k(\mathfrak{sl}_3)$ at admissible levels.
- Non-admissible levels (e.g. $k = -1$).
- Apply method to other minimal reductions, super-algebras ....